General Relation between $\vec{E}$ and $\vec{H}$

It can be shown that for any uniform plane wave traveling in an arbitrary direction shown with unit vector $\hat{k}$, $\vec{\tilde{E}}$ and $\vec{\tilde{H}}$ are:

$$\boxed{\begin{aligned} \vec{\tilde{H}} &= \frac{1}{\eta}\, \hat{k} \times \vec{\tilde{E}} \\ \vec{\tilde{E}} &= -\eta\, \hat{k} \times \vec{\tilde{H}} \end{aligned}}$$

valid for both lossless and lossy media, but η will be different.

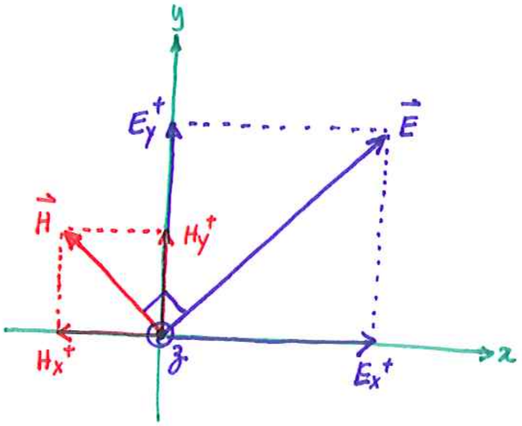
For example for a planewave of $\vec{\tilde{E}} = \hat{x}\, \tilde{E}_x^+(z) + \hat{y}\, \tilde{E}_y^+(z)$, the associated $\vec{\tilde{H}}$ is:

$$\vec{\tilde{H}} = \hat{x}\, \tilde{H}_x^+(z) + \hat{y}\, \tilde{H}_y^+(z)$$

Applying the above equation we have:

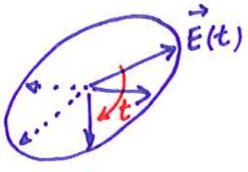
$$\vec{\tilde{H}} = \frac{1}{\eta}\, \hat{z} \times \vec{\tilde{E}} = -\hat{x}\, \frac{\tilde{E}_y^+(z)}{\eta} + \hat{y}\, \frac{\tilde{E}_x^+(z)}{\eta}$$

So we have: $\tilde{H}_x^+(z) = -\dfrac{\tilde{E}_y^+}{\eta}$ and $\tilde{H}_y^+(z) = \dfrac{\tilde{E}_x^+(z)}{\eta}$

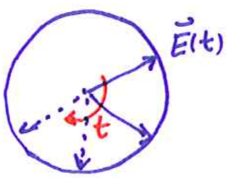


Wave Polarization

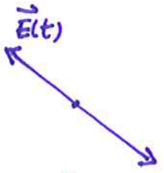
The polarization of a uniform plane wave describes the shape and locus of the tip of the $\vec{E}$ in the plane orthogonal to the direction of propagation at a given point in space as a function of time. Generally the locus is on an ellipse, which is called elliptically polarized. However, in some conditions it may change to a circle or a segment of a line, which is called circular or linear polarization.



Elliptical



Circular



linear

Since for a plane wave the z-component is zero, we can write:

$$\tilde{E}(z) = \hat{x}\tilde{E}_x(z) + \hat{y}\tilde{E}_y(z)$$

$$= \hat{x} E_{x_0} e^{-jkz} + \hat{y} E_{y_0} e^{-jkz}$$

we drop the e^{+jkz} terms for simplicity, or assume the wave is travelling in positive z direction. E_{x_0} and E_{y_0} are generally complex, so they have a magnitude and a phase. The phase is measured to a reference that can be $(z,t)=(0,0)$ or anything else. The polarization depends on the phase of E_{y_0} relative to that of E_{x_0} (not their absolute value). For simplicity we take the phase of E_{x_0} as the reference so $\angle E_{x_0} = 0$, and denote the phase of E_{y_0} as δ. So let's define:

$$\begin{cases} E_{x_0} = a_x & \rightarrow a_x = |E_{x_0}| \\ E_{y_0} = a_y e^{j\delta} & \rightarrow a_y = |E_{y_0}| \end{cases}$$

$$\Rightarrow \tilde{E}(z) = (\hat{x} a_x + \hat{y} a_y e^{j\delta}) e^{-jkz} \rightarrow \vec{E}(z,t) = \hat{x} a_x \cos(\omega t - kz) + \hat{y} a_y \cos(\omega t - kz + \delta)$$

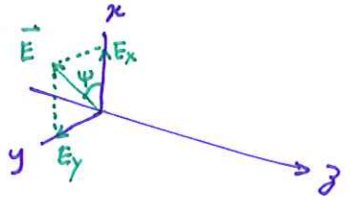
So for the intensity and direction of $\vec{E}(z,t)$, we have:

Intensity: $$|\vec{E}(z,t)| = [E_x^2(z,t) + E_y^2(z,t)]^{1/2}$$

$$= [a_x^2 \cos^2(\omega t - kz) + a_y^2 \cos^2(\omega t - kz + \delta)]^{1/2}$$

for the direction, we look at the angle of $\vec{E}$ with respect to x axis:

$$\psi(z,t) \triangleq \tan^{-1}\left(\frac{E_y(z,t)}{E_x(z,t)}\right)$$

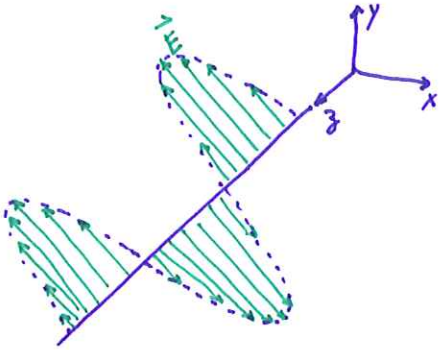


- Note that $\vec{E}$ is in the x-y plane with no component in z-direction (TEM).

Linear Polarization

$E_x(z,t)$ and $E_y(z,t)$ are in phase ($\delta = 0$) or out of phase ($\delta = \pi$).

⇒ Tip of $E(z=0,t)$ traces a straight line in the x-y plane.

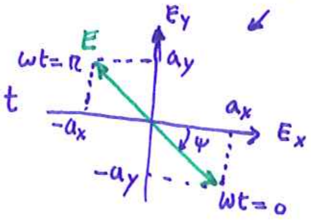


* At $z=0$ and $\delta=0$:

$$E(0,t) = (\hat{x} a_x + \hat{y} a_y)\cos\omega t \quad \text{(in-phase)}$$

* At $z=0$ and $\delta=\pi$:

$$E(0,t) = (\hat{x} a_x - \hat{y} a_y)\cos\omega t \quad \text{(out-of-phase)}$$



For the in phase:

$|E(0,t)| = [a_x^2 + a_y^2]^{1/2}\cos\omega t$ which changes with t

$\psi = \tan^{-1}\left(\frac{a_y}{a_x}\right)$ which is constant

For the out of phase:

$|E(0,t)| = [a_x^2 + a_y^2]^{1/2}\cos\omega t$ which changes with time

$\psi = \tan^{-1}\left(\frac{-a_y}{a_x}\right)$ which is constant

Circular Polarization

Magnitude of $\vec{E}_x$ and $\vec{E}_y$ are equal but the phase difference $\delta = \pm\frac{\pi}{2}$

when $\delta = \frac{\pi}{2}$ left-handed circular

when $\delta = -\frac{\pi}{2}$ right-handed circular

Left-handed Circular (LHC) polarization:

$$a_x = a_y = a \quad \text{and} \quad \underline{\underline{\delta = \frac{\pi}{2}}} \Rightarrow \tilde{E}(z) = (\hat{x}a + \hat{y}ae^{j\pi/2})e^{-jkz}$$

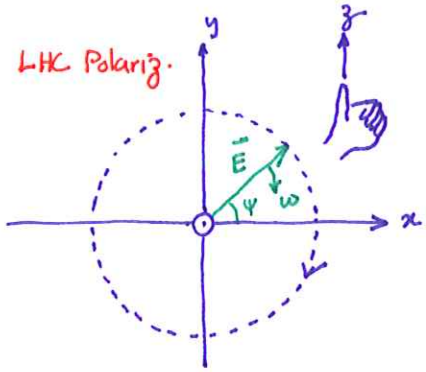
$$\vec{E}(z,t) = \hat{x}a\cos(\omega t - kz) + \hat{y}a\cos(\omega t - kz + \pi/2)$$

$$\rightarrow \vec{E}(z,t) = \hat{x}a\cos(\omega t - kz) - \hat{y}a\sin(\omega t - kz)$$

$$\rightarrow |\vec{E}(z,t)| = \left[a^2\cos^2(\omega t - kz) + a^2\sin^2(\omega t - kz)\right]^{1/2} = a \rightarrow \text{constant}$$

$$\psi(z,t) = \tan^{-1}\left(\frac{E_y}{E_x}\right) = \tan^{-1}\left(\frac{-a\sin(\omega t - kz)}{a\cos(\omega t - kz)}\right] = \tan^{-1}(-\tan(\omega t - kz)) = -(\omega t - kz)$$

→ ψ rotates to left (clockwise)



LHC Polariz.

Right-handed Circular (RHC) Polarization:

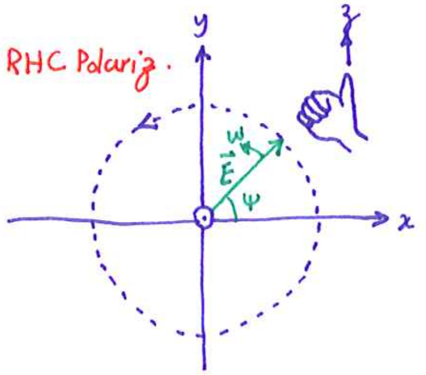
$$a_x = a_y = a \quad \text{and} \quad \underline{\underline{\delta = -\frac{\pi}{2}}} \Rightarrow \tilde{E}(z) = (\hat{x}a + \hat{y}ae^{-j\pi/2})e^{-jkz}$$

$$\vec{E}(z,t) = \hat{x}a(\cos\omega t - kz) + \hat{y}a\cos(\omega t - kz - \frac{\pi}{2})$$

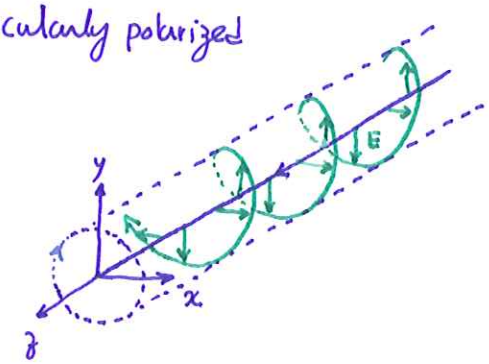
$$= \hat{x}a\cos(\omega t - kz) + \hat{y}a\sin(\omega t - kz)$$

$$\rightarrow |\vec{E}(z,t)| = a \rightarrow \text{constant}$$

$$\psi(z,t) = \tan^{-1}\left(\frac{E_y}{E_x}\right) = \omega t - kz$$ → rotates to right (counter cw) RHC Polariz.

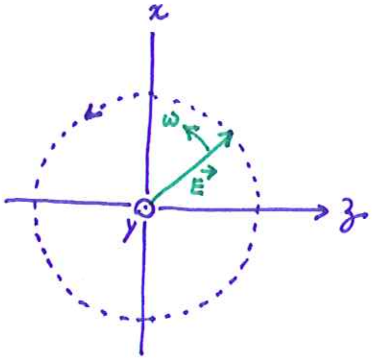


Left-hand circularly polarized wave:



Example

A RHC polarized plane wave is traveling in +y-direction. The modulus of E is $3 \frac{mV}{m}$ and $\varepsilon = 4\varepsilon_0$, $\mu = \mu_0$, and $\sigma = 0$ in the medium. Also $f = 100\,MHz$. Obtain expressions for $E(y,t)$ and $H(y,t)$.



$$\tilde{E}(y) = \hat{x}\tilde{E}_x + \hat{z}\tilde{E}_z$$

$$= \hat{x} a e^{-j\pi/2} e^{-jky} + \hat{z} a e^{-jky}$$

$$= (-\hat{x}j + \hat{z})\, 3e^{-jky}$$

$$\tilde{H}(y) = \frac{1}{\eta}\hat{y} \times \tilde{E}(y) = \frac{1}{\eta}\hat{y} \times (-\hat{x}j + \hat{z})3e^{-jky}$$

$$= \frac{3}{\eta}(\hat{z}j + \hat{x})e^{-jky}$$

$$\eta = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{\eta_0}{\sqrt{\varepsilon_r}} \cong \frac{120\pi}{\sqrt{4}} = 60\pi\ (\Omega)$$

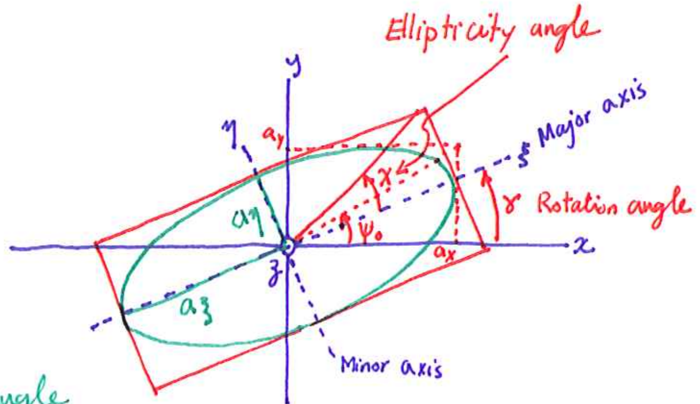
$$k = \omega\sqrt{\varepsilon\mu} = \frac{\omega\sqrt{\varepsilon_r}}{c} = \frac{2\pi \times 10^8 \times \sqrt{4}}{3 \times 10^8} = \frac{4}{3}\pi\ (rad/m) \quad ; \omega = 2\pi \times 10^8\ (rad/s)$$

$$\rightarrow \vec{E}(y,t) = 3\hat{x}\sin(\omega t - ky) + 3\hat{z}\cos(\omega t - ky) \quad (mV/m)$$

$$\vec{H}(y,t) = \frac{1}{20\pi}\left[\hat{x}\cos(\omega t - ky) - \hat{z}\sin(\omega t - ky)\right] \quad (mA/m)$$

Elliptical Polarization

In the most general case $a_x \neq 0$, $a_y \neq 0$, $\delta \neq 0$, the tip of $\vec{E}$ traces a ellipse in xy plane. This wave is called elliptically polarized.



Rotation angle: $-\frac{\pi}{2} \leq \gamma \leq \frac{\pi}{2}$

Ellipticity angle: $\frac{-\pi}{4} \leq \chi \leq \frac{\pi}{4}$; $\tan\chi = \pm\frac{a_\eta}{a_\xi} = \pm\frac{1}{R}$

Axial Ratio: $1 < R < \infty$ $\begin{cases} R = 1 \rightarrow \text{Circular pol.} \\ R = \infty \rightarrow \text{Linear pol.} \end{cases}$

$\begin{cases} \text{Left handed rotation}: & \chi > 0 \\ \text{Right handed rotation}: & \chi < 0 \end{cases}$

ψ_0 is Auxiliary angle

γ and χ are related to a_x, a_y, and δ by:

γ: rotation angle

χ: ellipticity angle → circle if 45°

In this course we don't derive these relations →

$$\tan 2\gamma = (\tan 2\psi_0)\cos\delta \qquad -\frac{\pi}{2} \le \gamma \le \frac{\pi}{2}$$

$$\sin 2\chi = (\sin 2\psi_0)\sin\delta \qquad -\frac{\pi}{4} \le \gamma \le \frac{\pi}{4}$$

where ψ_0 is an auxiliary angle: $\tan\psi_0 = \frac{a_y}{a_x}$ $\quad 0 \le \psi_0 \le \frac{\pi}{2}$

Always: $0 \le \psi_0 \le 90°$ because a_x and a_y are always +.

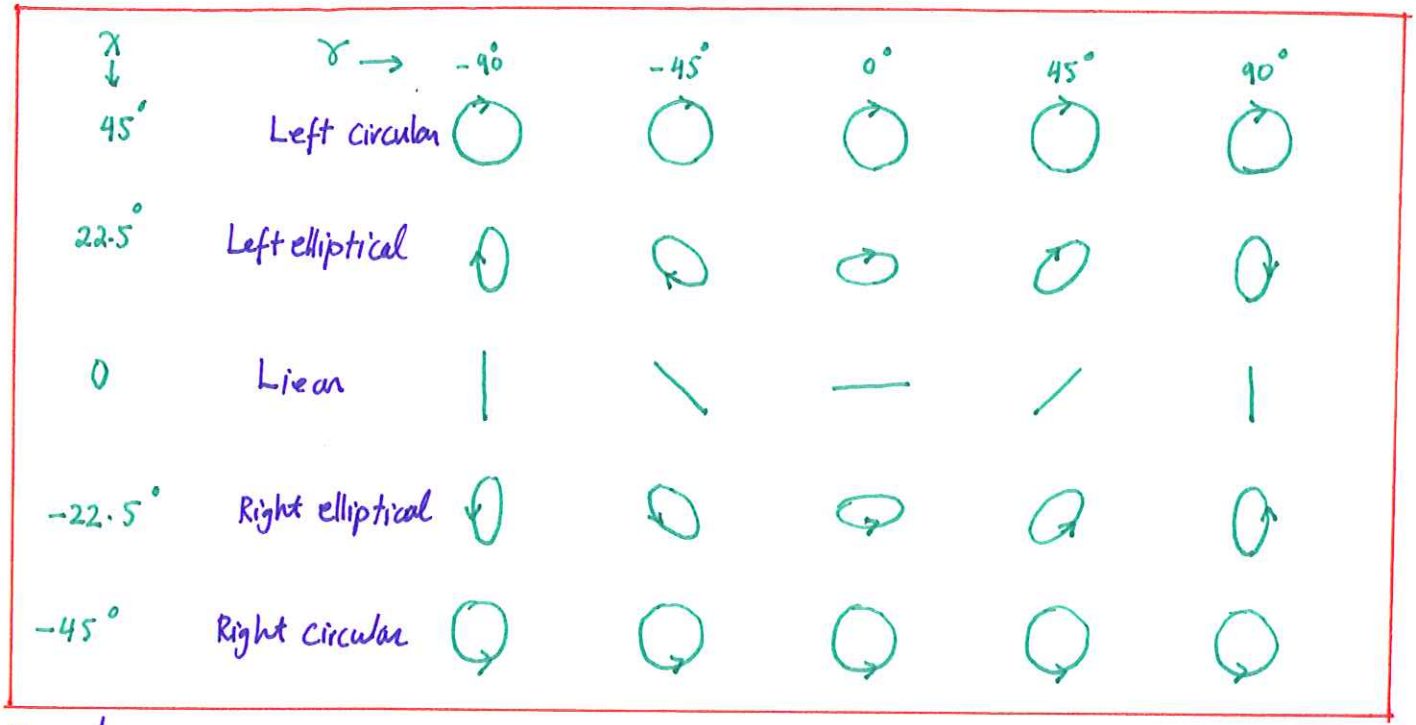
If $\chi = \pm 45° \to \tan\psi = \pm 1 \to R = 1 \to$ Circular polarization

If $\chi = 0 \to \tan\psi = 0 \to R = \infty \to$ Linearly polarized

Phase difference

for rotation angle: If $\cos\delta > 0 \to \tan 2\gamma > 0 \to \gamma > 0$

If $\cos\delta < 0 \to \tan 2\gamma < 0 \to \gamma < 0$



Example what is the polarization state of a plane wave with $\vec{E}(z,t) = \hat{x}\,3\cos(\omega t - kz + 30°) - \hat{y}\,4\sin(\omega t - kz + 45°)$

Solution: $\vec{E}(z,t) = \hat{x}\,3\cos(\omega t - kz + 30°) - \hat{y}\,4\cos(\omega t - kz + \overbrace{45° - 90°}^{-45°})$

$\to \tilde{E}(z) = \hat{x}\,3e^{j30}e^{-jkz} - \hat{y}\,4e^{-j45}e^{-jkz} = \hat{x}\,3e^{j30}e^{-jkz} + \hat{y}\,4e^{-j45}e^{j180}e^{-jkz}$

$\tilde{E}(z) = \hat{x}\,3e^{j30}e^{-jkz} + \hat{y}\,4e^{j135}e^{-jkz}$ $\to \delta_x = 30°$ and $\delta_y = 135 \to \delta = \delta_y - \delta_x = 105°$

→ The auxiliary angle $\psi_0 = \tan^{-1}\left(\frac{a_y}{a_x}\right) = \tan^{-1}\left(\frac{4}{3}\right) = 53.1°$

$\tan 2\gamma = (\tan 2\psi_0)\cos\delta = \tan(2\times 53.1)\cos(105) = 0.89 \to \gamma = 20.8°$ and $\gamma = -69.2°$.

Since $\cos\delta = \cos(105°) < 0 \to \gamma < 0 \to \gamma = -69.2°$ is only acceptable.

$$\sin 2\chi = (\sin(2\psi_0))\sin\delta$$

χ: ellipticity angle

$$= \sin 106.2^\circ \sin 105^\circ = 0.93 \rightarrow \chi = 34^\circ$$

Since $\chi \neq 45$ or 0 => elliptically polarized

Since $\chi > 0$ => left handed.

Plane-Wave Propagation in Lossy Media

We start with wave equation: $\nabla^2\tilde{E} - \gamma^2\tilde{E} = 0$

$$\text{where } \gamma^2 = -\omega^2\mu\varepsilon_c = -\omega^2\mu\underbrace{(\overbrace{\underset{\downarrow\ \varepsilon}{\varepsilon'} - j\underset{\downarrow\ \frac{\sigma}{\omega}}{\varepsilon''}}^{\varepsilon_c})}_{\downarrow}$$

$$\rightarrow \gamma \text{ is complex: } \gamma = \underset{\swarrow\ \text{Attenuation Constant}}{\alpha} + j\underset{\searrow\ \text{Phase constant}}{\beta}$$

$$\gamma^2 = (\alpha + j\beta)^2 = (\alpha^2 - \beta^2) + j2\alpha\beta = -\omega^2\mu\varepsilon' + j\omega^2\mu\varepsilon''$$

$$\rightarrow \begin{cases} \alpha^2 - \beta^2 = -\omega^2\mu\varepsilon' \\ 2\alpha\beta = \omega^2\mu\varepsilon'' \end{cases} \rightarrow \boxed{\begin{cases} \alpha = \omega\left\{\frac{\mu\varepsilon'}{2}\left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} - 1\right]\right\}^{1/2} & \text{Np/m} \\ \beta = \omega\left\{\frac{\mu\varepsilon'}{2}\left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} + 1\right]\right\}^{1/2} & \text{rad/m} \end{cases}}$$

For a uniform plane wave with $\underline{\tilde{E} = \hat{x}\tilde{E}_x(z)}$ traveling in $+z$-direction we have:

$$\nabla^2\tilde{E} - \gamma^2\tilde{E} = \frac{d^2\tilde{E}_x(z)}{dz^2} - \gamma^2\tilde{E}_x(z) = 0 \rightarrow \tilde{E}(z) = \hat{x}E_{x_0}e^{-\gamma z} = \hat{x}E_{x_0}e^{-\alpha z}e^{-j\beta z}$$

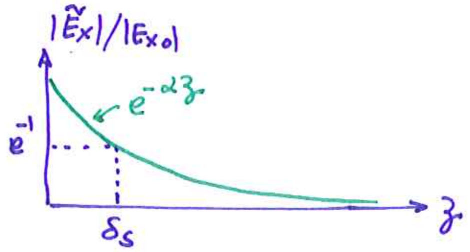
For $\tilde{H}$ we have: $\tilde{H} = \dfrac{(\hat{k}\times\tilde{E})}{\eta_c} = \dfrac{1}{\eta_c}\hat{z}\times\hat{x}E_{x_0}e^{-\gamma z} = \hat{y}\dfrac{E_{x_0}}{\eta_c}e^{-\alpha z}e^{-j\beta z}$

$$\text{where } \eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon' - j\varepsilon''}} = \sqrt{\frac{\mu}{\varepsilon'(1 - j\frac{\varepsilon''}{\varepsilon'})}} \rightarrow \boxed{\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon'}}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2}}$$

Note that $\tilde{H}$ has no longer same phase as $\tilde{E}$. Also note that magnitude of $\tilde{E}$ is reducing with z exponentially by attenuation constant α:

$$|\tilde{E}_x(z)| = |E_{x_0}e^{-\alpha z}e^{-j\beta z}| = |E_{x_0}|e^{-\alpha z}$$

Magnitude of $\tilde{H}$ also reduces with $e^{-\alpha z}$. The wave magnitude decreases by a factor of $e^{-1} \simeq 0.37$ through a distance $z = \delta_s$:



$$e^{-\alpha z} = e^{-1} \rightarrow z = \frac{1}{\alpha} \rightarrow \boxed{\delta_s = \frac{1}{\alpha}\ (\text{m})}$$ skin depth

The skin depth indicates how well an EM wave can penetrate into a conducting medium:

In perfect dielectric: $\sigma = 0 \rightarrow \alpha = 0 \rightarrow \delta_s = \infty$ So in free space EM can travel infinitely with no loss

In perfect Conductor: $\sigma = \infty \rightarrow \alpha = \infty \rightarrow \delta_s = 0$ So EM cannot penetrate into a good conductor.

In a coaxial cable the outer conductor shields the cable from outside EM wave as well as leakage of the wave from inside to outside.

$\frac{\varepsilon''}{\varepsilon'} \ll 1 \rightarrow$ low-loss dielectric (say $\frac{\varepsilon''}{\varepsilon'} < 10^{-2}$)

$\frac{\varepsilon''}{\varepsilon'} \gg 1 \rightarrow$ good conductor (say $\frac{\varepsilon''}{\varepsilon'} > 10^{2}$)

$10^{-2} < \frac{\varepsilon''}{\varepsilon'} < 10^{2} \rightarrow$ quasi-conductor

Low loss dielectric

we had $\gamma = j\omega\sqrt{\mu \varepsilon_c} = j\omega\sqrt{\mu\varepsilon'}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{1/2}$

if $|x| \ll 1$ we can approximate: $(1-x)^{1/2} \simeq 1 - \frac{x}{2}$

So if $\frac{\varepsilon''}{\varepsilon'} \ll 1 \Rightarrow \gamma \cong j\omega\sqrt{\mu\varepsilon'}\left(1 - j\frac{\varepsilon''}{2\varepsilon'}\right) = \alpha + j\beta \rightarrow$

$$\boxed{\begin{aligned} \alpha &\cong \frac{\omega\varepsilon''}{2}\sqrt{\frac{\mu}{\varepsilon'}} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} \quad (\text{Np/m}) \\ \beta &\cong \omega\sqrt{\mu\varepsilon'} = \omega\sqrt{\mu\varepsilon} \quad (\text{rad/m}) \end{aligned}}$$ Low loss dielectric

We can also approximate the intrinsic impedance η_c:

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon'}}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2} \simeq \sqrt{\frac{\mu}{\varepsilon'}}\left(1+j\frac{\varepsilon''}{2\varepsilon'}\right) = \sqrt{\frac{\mu}{\varepsilon}}\left(1+j\frac{\sigma}{2\omega\varepsilon}\right)$$

For the case $\frac{\varepsilon''}{\varepsilon'} < 10^{-2}$ we may further approximate this and ignore $\frac{\varepsilon''}{\varepsilon'}$ →

$$\boxed{\eta_c \cong \sqrt{\frac{\mu}{\varepsilon}}}$$ Low-loss dielectric

Good Conductor

For a good conductor $\frac{\varepsilon''}{\varepsilon'} > 10^2$. We can approximate α, β, and η_c as:

$$\alpha = \omega\left\{\frac{\mu\varepsilon'}{2}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2}-1\right]\right\}^{1/2} \simeq \omega\left\{\frac{\mu\varepsilon'}{2}\left(\frac{\varepsilon''}{\varepsilon'}\right)\right\}^{1/2} = \omega\sqrt{\frac{\mu\varepsilon''}{2}} = \omega\sqrt{\frac{\mu\sigma}{2\omega}}$$

$$\rightarrow \boxed{\alpha \cong \sqrt{\tfrac{1}{2}\mu\sigma\omega} = \sqrt{\pi f\mu\sigma}}$$ Good conductor

$$\boxed{\beta = \alpha \cong \sqrt{\pi f\mu\sigma}}$$ Good conductor

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon'}}\left(1-j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2} \cong \sqrt{\frac{\mu}{\varepsilon'}}\sqrt{\frac{1}{-j\frac{\varepsilon''}{\varepsilon'}}} = \sqrt{j\frac{\mu}{\varepsilon''}}$$

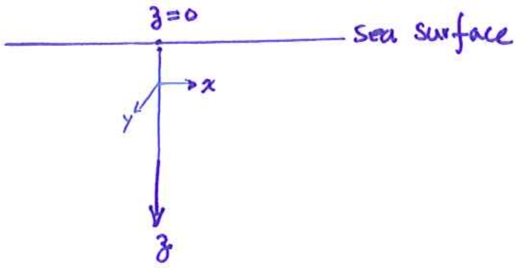
Since $\sqrt{j} = \frac{1}{\sqrt{2}}(1+j) \Longrightarrow \eta_c \cong \frac{1}{\sqrt{2}}(1+j)\sqrt{\frac{\mu}{\varepsilon''}} = (1+j)\sqrt{\frac{\mu\omega}{2\sigma}} = (1+j)\sqrt{\frac{\pi f\mu}{\sigma}}$

$$\sqrt{\pi f\mu} = \frac{\alpha}{\sqrt{\sigma}} \Rightarrow \boxed{\eta_c \cong (1+j)\frac{\alpha}{\sigma}}$$ Good conductor

Note: for perfect conductors $\sigma = \infty \rightarrow \alpha = \beta = \infty$ and $\eta_c = 0$ → Similar to a short circuit.

Example Plane wave in Seawater:

Consider a plane wave traveling deep into seawater as shown in picture. for seawater



$\varepsilon_r = 80$

$\mu_r = 1$

$\sigma = 4$ S/m

If the magnetic field at $z=0$ is $\underline{\vec{H}(0,t) = \hat{y}\,100\cos(2\pi\times10^{3}t+15^\circ)}$ $\left(\frac{mA}{m}\right)$

(a) Determine $E(z,t)$ and $H(z,t)$

(b) Determine the depth at which amplitude of $\vec{E}$ is 1% of its value at $z=0$.

Solution:

$\tilde{\vec{E}}(z) = \hat{x}E_{x_0}e^{-\alpha z}e^{-j\beta z}$ because $\vec{H}$ is in $\hat{y}$ direction.

$\tilde{E} = -\eta\,\hat{z}\times\tilde{H} = -\eta\,\overbrace{\hat{z}\times\hat{y}}^{-\hat{x}}\,\tilde{H}$

$\tilde{H}(z) = \hat{y}\,\frac{E_{x_0}}{\eta_c}e^{-\alpha z}e^{-j\beta z}$

we need to find α, β, and η_c. for that we first need $\frac{\varepsilon''}{\varepsilon'}$:

$$\frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{\omega\varepsilon_r\varepsilon_0} = \frac{4}{2\pi\times10^{3}\times80\times(10^{-9}/36\pi)} = 9\times10^{5}$$

Since $\frac{\varepsilon''}{\varepsilon'} \gg 1 \rightarrow$ seawater is a good conductor at 1 kHz $(\omega = 2\pi\times10^{3} \rightarrow f = 10^{3}\,\text{Hz} = 1\,\text{kHz})$

$\rightarrow \underline{\alpha = \sqrt{\pi f\mu\sigma}} = \sqrt{\pi\times10^{3}\times4\pi\times10^{-7}\times4} = 0.126\,(\text{Np/m})$ $\quad\beta = \alpha = 0.126\,\frac{\text{rad}}{\text{m}}$

$\underline{\eta_c = (1+j)\frac{\alpha}{\sigma}} = \sqrt{2}\,e^{j\pi/4}\,\frac{0.126}{4} = 0.044\,e^{j\pi/4}\ (\Omega) \quad \rightarrow \frac{1}{\eta_c} = 22.5\,e^{-j\pi/4}$

If we take $\underline{\tilde{E}_x(z) = \hat{x}\,|E_{x_0}|e^{j\phi_0}e^{-\alpha z}e^{-j\beta z}} \rightarrow$

$$\begin{cases} \vec{E}(z,t) = \hat{x}\,|E_{x_0}|\,e^{-0.126z}\cos(2\pi\times10^{3}t - 0.126z + \overset{?}{\underline{\phi_0}}) \quad \left(\frac{V}{m}\right) \\ \vec{H}(z,t) = \hat{y}\,22.5\,|E_{x_0}|\,e^{-0.126z}\cos(2\pi\times10^{3}t - 0.126z + \overset{?}{\underline{\phi_0}} - 45^\circ) \quad \left(\frac{A}{m}\right) \end{cases}$$

At $z=0$ $\quad \vec{H}(0,t) = \hat{y}\,22.5|E_{x_0}|\cos(2\pi\times10^{3}t + \phi_0 - 45) = \hat{y}\,100\cos(2\pi\times10^{3}t + 15)\ \left(\frac{mA}{m}\right)$

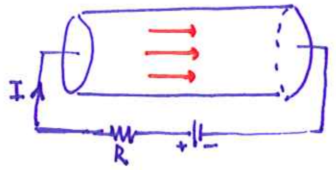
$$\rightarrow \begin{cases} 22.5|E_{x_0}| = 100\times10^{-3} \rightarrow |E_{x_0}| = 4.44\ \left(\frac{mV}{m}\right) \\ \phi_0 - 45^\circ = 15^\circ \rightarrow \phi_0 = 60^\circ \end{cases}$$

$$\rightarrow \begin{cases} E(z,t)=\hat{x}\,4.44\,e^{-0.126z}\cos(2\pi\times10^{3}t-0.126z+60^\circ) & (\frac{mV}{m}) \\ H(z,t)=\hat{y}\,100\,e^{-0.126z}\cos(2\pi\times10^{3}t-0.126z+15^\circ) & (\frac{mA}{m}) \end{cases}$$

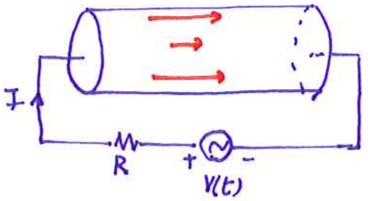
(b) $0.01=e^{-0.126z} \rightarrow z=\frac{1}{-0.126}\ln(0.01)=36m$

Current Flow in a good Conductor

In DC current is uniform in the conductor

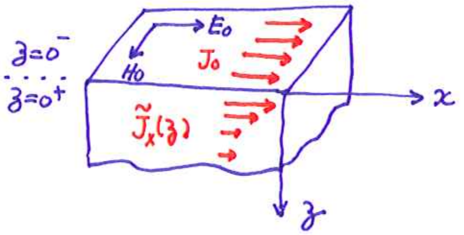


In AC current density is maximum close to the surface.



At very high frequency, the current is almost zero in the center and most of the current goes in a thin layer in the outer region of the wire.

Consider a semi-infinite solid as shown here:



If at $z=0^-$, $\vec{\tilde{E}}=\hat{x}E_0$ and $\vec{\tilde{H}}=\hat{y}\frac{1}{\eta}E_0$:

Tangential part of E is continous $\Rightarrow E(z=0^+)=E(z=0^-)$

$$\rightarrow \vec{\tilde{E}}(0^+)=\hat{x}E_0 \rightarrow \begin{cases} \vec{\tilde{E}}(z)=\hat{x}E_0e^{-\alpha z}e^{-j\beta z} \\ \tilde{H}(z)=\hat{y}\frac{E_0}{\eta_c}e^{-\alpha z}e^{-j\beta z} \end{cases}$$

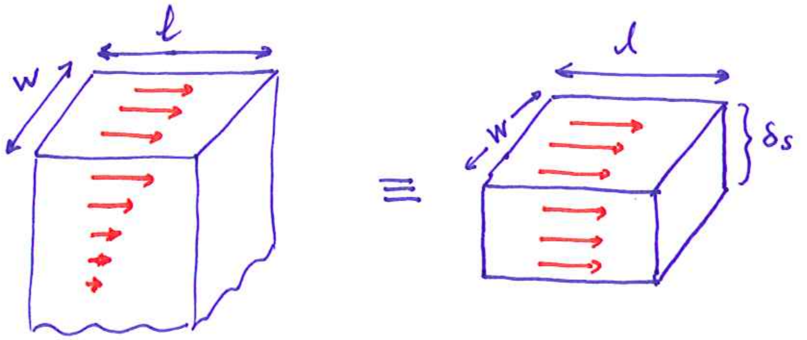
To get $\vec{J}$: $\underline{\vec{J}=\sigma\vec{E}}$ $\Rightarrow \vec{\tilde{J}}(z)=\hat{x}\tilde{J}_x(z)=\hat{x}\underbrace{\sigma E_0}_{J_0}e^{-\alpha z}e^{-j\beta z}=\hat{x}J_0e^{-\alpha z}e^{-j\beta z}$

For a good conductor $\alpha\cong\beta$. Also, we know the skin depth $\delta_s=\frac{1}{\alpha}\Rightarrow$

$$\underline{\vec{\tilde{J}}_x(z)}=\hat{x}\,J_0\,e^{-(1+j)z/\delta_s} \quad (A/m^2)$$

The total current is: $\underline{\tilde{I}}=w\int_0^\infty \tilde{J}_x(z)dz=w\int_0^\infty J_0e^{-(1+j)z/\delta_s}dz=\frac{\boxed{J_0w\delta_s}}{1+j}$ (A)

This is like uniform current density in thin surface section of δ_s thick.



the voltage drop across the length of l is: $\tilde{V} = E_0 l = \frac{J_0}{\sigma} l$

The impedance of the semiinfite slab of width w, length l, and depth $d=\infty$ is:

$$Z = \frac{\tilde{V}}{\tilde{I}} = \frac{1+j}{\sigma \delta_s}\frac{l}{w} \equiv Z_s \frac{l}{w}$$

↓ Internal or Surfae impedance

$$\boxed{Z_s = \frac{1+j}{\sigma \delta_s} \quad (\Omega)}$$

$$Z_s = R_s + j\omega L_s$$

$$\delta_s = \frac{1}{\alpha} \simeq \frac{1}{\sqrt{\pi f \mu \sigma}}$$

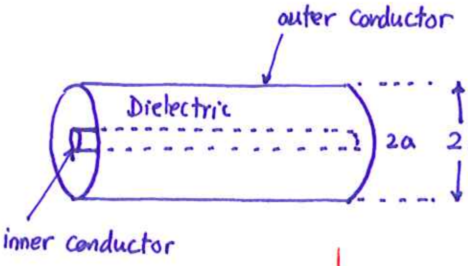
$\Rightarrow$

$$R_s = \frac{1}{\sigma \delta_s} = \sqrt{\frac{\pi f \mu}{\sigma}} \quad (\Omega)$$

$$L_s = \frac{1}{\omega \sigma \delta_s} = \frac{1}{2}\sqrt{\frac{\mu}{\pi f \sigma}} \quad (H)$$

The AC resistance is: $R = R_s \frac{l}{w} = \frac{l}{\underbrace{\sigma \delta_s w}_{A}} \quad (\Omega)$

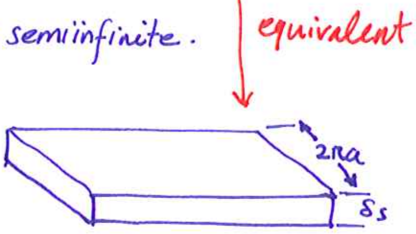
Coaxial Cable



The conductors are made of Copper with $\sigma = 5.8 \times 10^7$ S/m

$$\delta_s = \frac{1}{\sqrt{\pi f \mu \sigma}} = 0.066\text{mm at } 1\text{MHz}$$

If $a > 5\delta_s$ or $a > 0.33$mm, its "depth" may be regarded as semiinfinite. So the current is limitted to a thin outer layer of δ_s thick with circumference $2\pi a$.



→ Resistance per unit length is: $R'_1 = \frac{R}{l} = \frac{1}{l} R_s \frac{l}{\underbrace{2\pi a}_{w}} = \frac{R_s}{2\pi a} \quad (\Omega/m)$

Similarly for the outer conductor with radius b:

$$R_2' = \frac{R_S}{2\pi b} \quad (\Omega/m)$$

The total ac resistance per unit length is:

$$R' = R_1' + R_2' = \frac{R_S}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right) \quad (\Omega/m)$$

Electromagnetic Power Density

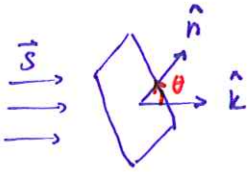
For any EM wave, the Poynting vector S is defined as:

$$\boxed{\vec{S} = \vec{E}\times\vec{H} \quad (W/m^2)}$$

$\vec{S}$ is in the direction of the wave $\hat{k}$. The total power through a surface A is:

$$P = \int_A \vec{S}\cdot\hat{n}\,dA \quad (W)$$

If n makes an angle θ with the plane : $P = SA\cos\theta$



Since $\vec{E}$ and $\vec{H}$ are a function of time, $\vec{S}$ is also a function of time.

The time average of power density Sav is given by:

$$\boxed{S_{av} = \frac{1}{2}\mathrm{Re}\left[\tilde{\vec{E}}\times\tilde{H}^*\right]} \left(\frac{W}{m^2}\right)$$

Plane wave in a Lossless Medium _ Average power density

General form: $\tilde{\vec{E}}(z) = \hat{x}\,\tilde{E}_x(z) + \hat{y}\,\tilde{E}_y(z) = (\hat{x}E_{x_0} + \hat{y}E_{y_0})e^{-jkz}$

$\Rightarrow \quad |\tilde{\vec{E}}| = (\tilde{E}\cdot\tilde{E}^*)^{1/2} = \left[|E_{x0}|^2 + |E_{y0}|^2\right]^{1/2}$

$\Rightarrow \quad \tilde{\vec{H}}(z) = \frac{1}{\eta}\,\hat{z}\times\tilde{E} = \frac{1}{\eta}(-\hat{x}E_{y0} + \hat{y}E_{x0})e^{-jkz}$

$$S_{av} = \frac{1}{2}\mathrm{Re}(\tilde{E}\times\tilde{H}^*) = \hat{z}\frac{1}{2\eta}\Big(\underbrace{|E_{x0}|^2}_{\text{power density for }(\tilde{E}_x,\,\tilde{H}_y)\text{ wave}} + \underbrace{|E_{y0}|^2}_{\text{power density for }(\tilde{E}_y,\,\tilde{H}_x)\text{ wave}}\Big) = \hat{z}\frac{|\tilde{E}|^2}{2\eta} \quad (W/m^2)$$

→ Polarization has no effect on Sav

Example Solar Power

Solar power: 1 kW/m² at earth's surface

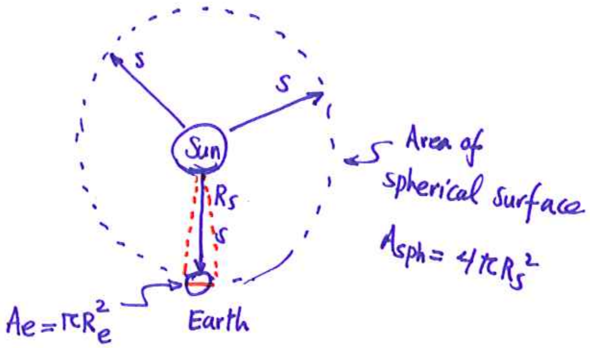
(a) find the total power radiated by the sun ?

(b) the total power intercepted by earth ?

(c) electric field of the power density incident upon Earth's surface assuming that all the solar illumination is at a single frequency.

The radius of Earth's orbit around the sun, R_s, is approximately 1.5×10^8 km, and Earth's mean radius $R_e \simeq 6380$ km.

Solution: (a)



$$P_{sun} = S_{av}(4\pi R_s^2) = 1\times10^3 \times 4\pi \times (1.5\times10^{11})^2$$
$$= 2.8\times10^{26}\ \text{W}$$

(b) The power intercepted by Earth's cross section

$$A_e = S_{av}(\pi R_e)^2 = 1\times10^3 \times \pi \times (6.38\times10^6)^2$$
$$= 1.28\times10^{17}\ \text{W}$$

(c) S_{av} is related to the magnitude of the electric field $|E_0|$ by :

$$S_{av} = \frac{|E_0|^2}{2\eta_0} \rightarrow |E_0| = \sqrt{2\eta_0 S_{av}} = \sqrt{2\times377\times10^3} = 870\ \text{(V/m)}$$

Plane Wave in a lossy Medium – Average Power density

$$\vec{\tilde{E}}(z) = \hat{x}\,\tilde{E}_x(z) + \hat{y}\,\tilde{E}_y(z)$$
$$= (\hat{x}E_{x0} + \hat{y}E_{y0})e^{-\alpha z}e^{-j\beta z}$$
$$\vec{\tilde{H}}(z) = \frac{1}{\eta_c}(-\hat{x}E_{y0} + \hat{y}E_{x0})e^{-\alpha z}e^{-j\beta z}$$

$$\vec{S}_{av}(z) = \frac{1}{2}\,\mathrm{Re}\left[\vec{\tilde{E}} \times \vec{\tilde{H}}^*\right]$$
$$= \frac{\hat{z}\,(|E_{x0}|^2 + |E_{y0}|^2)}{2}\,\underline{e^{-2\alpha z}}\,\mathrm{Re}\left(\frac{1}{\eta_c^*}\right)$$

$$\eta_c = |\eta_c|e^{j\theta_\eta} \Rightarrow \boxed{S_{av}(z) = \hat{z}\,\frac{|E_0|^2}{2|\eta_c|}\,e^{-2\alpha z}\cos\theta_\eta \quad \left(\frac{\text{W}}{\text{m}^2}\right)}$$

So whereas $\vec{\tilde{E}}$ and $\vec{\tilde{H}}$ decay by $e^{-\alpha z}$, S_{av} decays by $e^{-2\alpha z}$

Decibel Scale for Power Ratios

The decibel (dB) scale is logarithmic and often used to express the ratio of two powers like P_1 and P_2 :

$$G = \frac{P_1}{P_2} \rightarrow \qquad G[dB] \triangleq 10 \log G = 10 \log \left(\frac{P_1}{P_2}\right) \quad (dB)$$

G		G[dB]
10^x	=	$10x$ dB
4	=	6 dB
2	=	3 dB
1	=	0 dB
0.5	=	−3 dB
0.25	=	−6 dB
0.1	=	−10 dB
10^{-3}	=	−30 dB

we can relate same unit for the voltage ratios:

$$P_1 = \frac{V_1^2}{R} \qquad P_2 = \frac{V_2^2}{R} \rightarrow \qquad \frac{P_1}{P_2} = \frac{V_1^2}{V_2^2} \rightarrow$$

$$G[dB] = 10 \log \left(\frac{P_1}{P_2}\right) = 10 \log \left(\frac{V_1}{V_2}\right)^2 = 20 \log \left(\overbrace{\frac{V_1}{V_2}}^{g}\right) = 20 \log (g)$$

$$\boxed{g[dB] \triangleq 20 \log (g)}$$

Note the scale factor for voltage (and also current) is 20 .

Attenuation rate:

$$A = 10 \log \left(\frac{S_{av}(z)}{S_{av}(0)}\right) = 10 \log \left(e^{-2\alpha z}\right) = -20 \alpha z \log e = -\overbrace{8.68 \alpha}^{\alpha [dB/m]} z.$$

$$= -\alpha \, [dB/m] \, z$$ where we defined $\alpha[dB/m] \triangleq 8.68 \alpha \; [Np/m]$

Also: $$A = 10 \log \frac{|E(z)|^2}{|E(0)|^2} = 20 \log \frac{|E(z)|}{|E(0)|}$$